§7.4 Basic Theory of Systems of First-Order Linear Equations

We have our general linear system

$$\vec{x}' = P(t)\vec{x} + \vec{g}(t)$$

where P is an $n \times n$ matrix. Continuity of P and $\vec{g}$ guarantees of solutions.

We will first consider the homogeneous equation

$$\vec{x}' = P(t)\vec{x}.$$

Later, we will consider the nonhomogeneous equation.

<u>Thm</u> (Principle of Superposition)

If $\vec{x}^{(1)}$ and $\vec{x}^{(2)}$ are solutions of $\vec{x}' = P(t)\vec{x}$, then $c_1\vec{x}^{(1)} + c_2\vec{x}^{(2)}$ is a solution for any constants c_1 and c_2.

<u>Remark</u> This result extends to $\vec{x}^{(1)}, \ldots, \vec{x}^{(k)}$ are solutions of $\vec{x}' = P(t)\vec{x}$, then

$$\vec{x} = c_1 \vec{x}^{(1)} + \cdots + c_k \vec{x}^{(k)}$$

is a solution.

i.e. every finite linear combination of solutions is a solution.

Consider the matrix

$$\mathbf{X}(t) = \begin{pmatrix} | & | & & | \\ \vec{x}^{(1)} & \vec{x}^{(2)} & \cdots & \vec{x}^{(n)} \\ | & | & & | \end{pmatrix}$$

where $\vec{x}^{(1)}, \ldots, \vec{x}^{(n)}$ are solutions. The columns of $\mathbf{X}(t)$ are linearly independent if and only if $\det \mathbf{X} \neq 0$. The determinant of $\mathbf{X}$ is the Wronskian of $\vec{x}^{(1)}, \ldots, \vec{x}^{(n)}$

and denoted $W[\vec{x}^{(1)}, \vec{x}^{(2)}, \ldots, \vec{x}^{(n)}]$.

<u>Thm</u> If $\vec{x}^{(1)}, \ldots, \vec{x}^{(n)}$ are linearly independent solutions, then the linear combination

$$\vec{x} = c_1 \vec{x}^{(1)}(t) + c_2 \vec{x}^{(2)}(t) + \cdots + c_n \vec{x}_n(t) \quad (*)$$

expresses each solution of the system $\vec{x}' = P(t)\vec{x}$ in exactly one way.

If $c_1, \ldots, c_n$ are thought of as arbitrary, then $(*)$ is called the general solution. A set of linearly independent solutions is called a fundamental set of solutions.

<u>Thm</u> (Abel's Theorem)

If $\vec{x}^{(1)}, \ldots, \vec{x}^{(n)}$ are solutions on

$\alpha < t < \beta$, then $W[\vec{x}^{(1)}, \dots, \vec{x}^{(n)}]$ is either identically zero or never vanishes.

$W[\vec{x}^{(1)}, \dots, \vec{x}^{(n)}](t) = 0$ for all $t \in (\alpha, \beta)$

<u>Thm</u> If $\vec{x} = \vec{u}(t) + i\vec{v}(t)$ is a complex-valued solution, then its real part $\vec{u}(t)$ and its imaginary part $\vec{v}(t)$ are also solutions.

<u>7.5 Homogeneous Linear Systems with Constant Coefficients</u>

Consider the system

$$\vec{x}' = A\vec{x}$$

where A is constant $n \times n$ matrix. The equilibria of the system are where $\vec{x}' = 0$.

i.e. where $A\vec{x} = 0$.

If $\det(A) \neq 0$, then $\vec{x} = 0$ is the only equilibria.

What about the stability of $\vec{x} = 0$?

When $n = 2$, we can visualize qualitative features in the x_1x_2-plane (phase plane).

Use $A\vec{x}$ to plot the direction field, and we can plot a number of solution curves (phase portrait).

Solving Systems of ODEs

Consider the system

$$\vec{x}' = A\vec{x}.$$

We look for solutions of the form

$$\vec{x} = \vec{\xi} e^{rt}.$$

Then,

$$\vec{\xi} e^{rt} r = A\vec{\xi} e^{rt}$$

$$A\vec{\xi} e^{rt} - \vec{\xi} e^{rt} r = 0$$

$$(A - rI)\vec{\xi} e^{rt} = 0$$

Since $e^{rt} \neq 0$, we must have that

$$(A - rI)\vec{\xi} = 0.$$

We see that r is an eigenvalue of A and $\vec{\xi}$ is the corresponding eigenvector.

<u>Ex</u> $\vec{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \vec{x}$

First, we find the eigenvalues and eigenvectors.

$$\det(A - rI) = 0$$

$$\det\left(\begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} - r\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = 0$$

$$\det\begin{pmatrix} 1-r & 1 \\ 4 & 1-r \end{pmatrix} = 0$$

$$(1-r)^2 - 4 = 0$$

$$r^2 - 2r - 3 = 0$$

$$(r-3)(r+1) = 0$$

$$r_1 = 3 \qquad r_2 = -1$$

Now, we find the associated eigenvectors.

$\underline{r_1 = 3}$ $\qquad \vec{\xi}_1 = \begin{pmatrix} \xi_{11} \\ \xi_{12} \end{pmatrix}$

$$(A - r_1 I)\vec{\xi}_1 = 0$$

$$\begin{pmatrix} 1-3 & 1 \\ 4 & 1-3 \end{pmatrix}\begin{pmatrix} \xi_{11} \\ \xi_{12} \end{pmatrix} = 0$$

$$\begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} \xi_{11} \\ \xi_{12} \end{pmatrix} = 0$$

$$-2\xi_{11} + \xi_{12} = 0$$

$$\xi_{12} = 2\xi_{11}$$

Therefore,

$$\vec{\xi}_1 = \begin{pmatrix} \xi_{11} \\ \xi_{12} \end{pmatrix} = \begin{pmatrix} \xi_{11} \\ 2\xi_{11} \end{pmatrix} = \xi_{11} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$\underline{r_2 = -1}$

$$(A - r_2 I)\vec{\xi}_2 = 0$$

$$\left(\begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} - \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right) \vec{\xi}_2 = 0$$

$$\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} \xi_{21} \\ \xi_{22} \end{pmatrix} = 0$$

$$2\xi_{21} + \xi_{22} = 0$$

$$\xi_{22} = -2\xi_{21}$$

Therefore,

$$\vec{\xi}_2 = \begin{pmatrix} \xi_{21} \\ \xi_{22} \end{pmatrix} = \begin{pmatrix} \xi_{21} \\ -2\xi_{21} \end{pmatrix} = \xi_{21} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Our solutions are

$$\vec{x}^{(1)}(t) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} \quad \text{and} \quad \vec{x}^{(2)}(t) = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}$$

The Wronskian of the solutions is

$$W[\vec{x}^{(1)}, \vec{x}^{(2)}](t) = \det \begin{pmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{pmatrix}$$

$$= -2e^{2t} - 2e^{2t}$$

$$= -4e^{2t}$$

$$\neq 0$$

So $\vec{x}^{(1)}$ and $\vec{x}^{(2)}$ form a fundamental set of solutions.

Thus,

$$\vec{x}(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}.$$

Drawing the Phase Plane

1. Determine equilibria
2. Determine eigenvalues and eigenvectors

$$\vec{x}' = A\vec{x} \qquad \text{eigen-pair } r, \vec{\xi}$$

$$A\vec{\xi} = r\vec{\xi}$$

3. Determine nullclines
   A nullcline is where $x_i' = 0$
4. Determine the sign of x_i' in regions

determined by the null clines

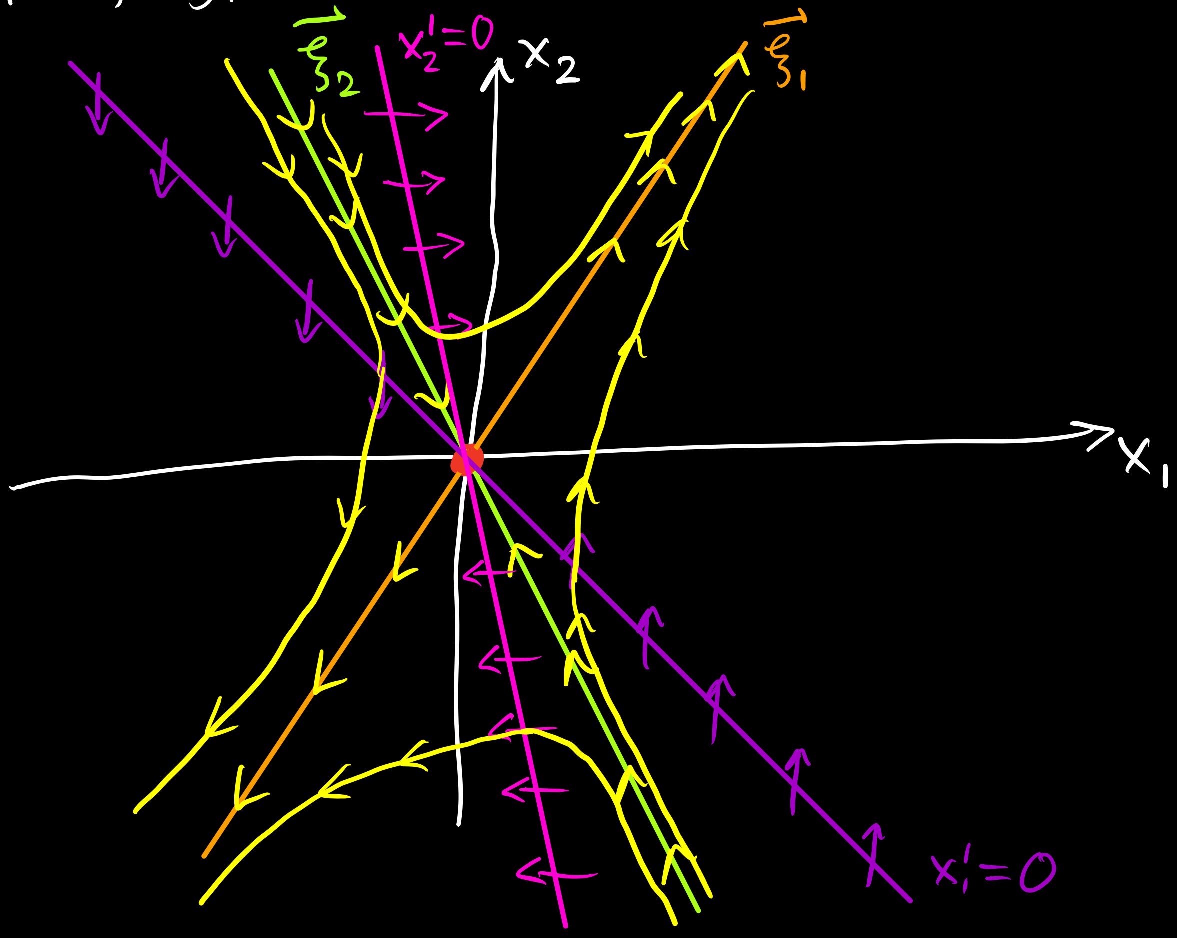
5. Draw sample trajectories.

Ex

$$\vec{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \vec{x}$$

We found the eigenpairs

$r_1 = 3$, $\vec{\xi}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $r_2 = -1$, $\vec{\xi}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.



Nullclines

$$x_1' = x_1 + x_2$$
$$0 = x_1 + x_2$$
$$x_2 = -x_1$$

$$x_2' = 4x_1 + x_2$$
$$0 = 4x_1 + x_2$$
$$x_2 = -4x_1$$

$$x_1' = 0$$
$$x_2' = 4x_1 + x_2$$
$$= -4x_2 + x_2$$
$$= -3x_2$$

$$x_2' = 0$$
$$x_1' = x_1 + x_2$$
$$= x_1 - 4x_1$$
$$= -3x_1$$

General nxn systems

1. Find eigenvectors and eigenvalues
   $r_1, \dots, r_n, \vec{\xi}_1, \dots, \vec{\xi}_n$

2. General Solution

$$\vec{x} = c_1 \vec{\xi}_1 e^{r_1 t} + \cdots + c_n \vec{\xi}_n e^{r_n t}$$

3. Use ICs to determine $c_1, \ldots, c_n$.

The eigenvalues are determined by the n^{th} degree polynomial equation

$$\det(A - \lambda I) = 0 .$$

Possible eigenvalues

1. real and distinct
2. Complex conjugate pairs
3. repeated roots

Ex

$$\vec{x}' = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \vec{x}$$

First, we find the eigenvalues,

$$\det\begin{pmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{pmatrix} = 0$$

$$-\lambda \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 1 \\ 1 & -\lambda \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 1 \\ -\lambda & 1 \end{vmatrix} = 0$$

$$-\lambda(\lambda^2 - 1) - (-\lambda - 1) + (1 + \lambda) = 0$$

$$-\lambda(\lambda+1)(\lambda-1) + (\lambda+1) + (\lambda+1) = 0$$

$$(\lambda+1)(-\lambda(\lambda-1) + 2) = 0$$

$$(\lambda+1)(-\lambda^2 + \lambda + 2) = 0$$

$$(\lambda+1)(-\lambda-1)(\lambda-2) = 0$$

$$\lambda_1 = -1, \ \lambda_2 = -1, \ \lambda_3 = 2$$

$\underline{\lambda_1 = -1}$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} v_{11} \\ v_{12} \\ v_{13} \end{pmatrix} = 0$$

$$v_{11} + v_{12} + v_{13} = 0$$

$$v_{11} = -v_{12} - v_{13}$$

$$\vec{v}_1 = \begin{pmatrix} v_{11} \\ v_{12} \\ v_{13} \end{pmatrix} = \begin{pmatrix} -v_{12} - v_{13} \\ v_{12} \\ v_{13} \end{pmatrix}$$

$$= \begin{pmatrix} -v_{12} \\ v_{12} \\ 0 \end{pmatrix} + \begin{pmatrix} -v_{13} \\ 0 \\ v_{13} \end{pmatrix}$$

$$= v_{12} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + v_{13} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Therefore, $\vec{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$\underline{\lambda_3 = 2}$

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} v_{31} \\ v_{32} \\ v_{33} \end{pmatrix} = 0$$

$$\begin{pmatrix} -2 & 1 & 1 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{pmatrix} \begin{pmatrix} v_{31} \\ v_{32} \\ v_{33} \end{pmatrix} = 0$$

$$\begin{pmatrix} -2 & 1 & 1 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_{31} \\ v_{32} \\ v_{33} \end{pmatrix} = 0$$

$$-3v_{32} + 3v_{33} = 0$$

$$v_{32} = v_{33}$$

$$-2v_{31} + v_{32} + v_{33} = 0$$

$$-2v_{31} = -v_{32} - v_{33}$$

$$= -2v_{33}$$

$$\vec{v}_3 = \begin{pmatrix} v_{31} \\ v_{32} \\ v_{33} \end{pmatrix} = \begin{pmatrix} v_{33} \\ v_{33} \\ v_{33} \end{pmatrix} = v_{33} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Therefore,

$$\vec{x}(t) = c_1 e^{-t} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + c_3 e^{2t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$